J. Appl. Math. & Informatics Vol. **44**(2026), No. 2, pp. 759 - 770
https://doi.org/10.14317/jami.2026.759

DERIVATION AND DISPERSION ANALYSIS OF COMPACT SCHEMES APPLIED IN DIFFERENTIAL EQUATIONS WITH THIRD DERIVATIVE

H.A.S. AL-DUJALY*, A. BAKHEET AND E.A. ALNUSSAIRY

Abstract. Simulations and approximations of higher-order derivatives are essential to describe many complex phenomena in engineering and physics. To ensure stability and accuracy, compact numerical approximations are employed to solve many differential equations with problems involving dispersive waves. This work introduces a new class of implicit compact finite difference schemes that systematically generate fourth-, sixth-, and eighth-order accurate approximations for the third derivative. To detect many types of schemes with different orders of accuracy, the process of solving several systems of equations is required. The desired accuracy for each scheme is verified by employing special polynomials in the schemes and obtaining errors as the remaining terms for these polynomials. In addition to gaining the accuracy and stability of introduced schemes, achieving high resolutions makes them popular candidates when dealing with hyperbolic problems. From Fourier analysis, dispersive errors for the derived schemes are examined. These errors clearly illustrate the ability of high order compact schemes in capturing high resolutions. Finally, schemes proposed from this work are applicable to many computational problems and can be expanded to solve some types of multi-dimensional problems.

AMS Mathematics Subject Classification : 65M06, 35K99, 65M12.
Key words and phrases : Dispersion errors, high accuracy, third derivative, wave number, compact approximation.

1. Introduction

Recently, when differential equations do not have analytical solutions, the use of numerical differentiation has become an indispensable technique in scientific computing and computational mathematics. In many scientific and engineering

---

Received January 8, 2026. Revised March 6, 2026. Accepted March 15, 2026. *Corresponding author.

disciplines, using differential equations to formulate mathematical models becomes essential for analysis and simulation [6]. In all approaches to establish numerical methods, studying the convergence and stability for these methods is necessary to detect their ability and accuracy, even the Iteration methods; see [10, 11] . There are many methods of constructing numerical differentiation by applying function values at a set of discrete points, and these methods are called finite difference methods because of using the finite differences between function values instead of the infinitesimal differences in the definition of the derivative. The main idea for deriving finite difference approximations relies on the Taylor series expansion of the function by approximating the values of its derivatives at a nearby point while the value of a function at a point is based on its value. New versions of finite difference methods have been modified and employed to solve my equation models, such as the Burgers equation [5], the vorticity transport equation [12, 14] and the helmholtz equation [20, 22] . In addition, many implicit Runge-Kutta methods are established and improved, especially multi step methods [8, 9, 15] . In general, when using a very small time step size in the simulation, the computational cost might be increased. To overcome these challenges, different types of implicit schemes have been established, such as compact schemes, which maintain more precision and stability compared to explicit schemes [1, 16, 17]. Various classes of compact schemes are derived according to the needs of study problems and applications such as capturing shock areas [1], avoiding negative dissipation [16] , employing Fourier analysis [4, 7, 21] and maintaining high resolutions [13] . In addition to the compact schemes of the first derivative [3], the methodology of the second derivative is also discussed [2]. Among other orders of derivatives, the third derivative is of particular significance in describing some physical phenomena, like the Korteweg-de Vries equation. Hence, the numerical approximation of the third derivative is essential to capture the behavior of many systems and to estimate the true derivative for these systems and their applications. Indeed, electing the appropriate scheme for approximating the third derivative impacts the stability of the solution for equations like the dispersive wave equation. To ensure stability for higher-order PDEs involving third derivatives, strict time step restrictions might be adhered when employing explicit finite difference schemes [18] .
In this work, a general class of third derivative is introduced, and the construction processes of several numerical approximations are indicated in section 2. From matching both sides of the proposed schemes, the order of accuracy for these schemes is verified in section 3. Section 4 contains the procedures for analyzing the derived schemes by calculating modified wavenumbers, and the dispersion errors are examined and compared with the exact wave number.

2. The Derivation Of The Third Derivative Schemes

High order numerical schemes for the third derivative are derived in this section. The points of the grid $x_0, x_1, \ldots, x_N$ along with the step size $h = x_k -$

x_{k-1} are considered so that $1 \leq k \leq N$. g_k''' represents the approximation of the third derivative of $g(x)$ in x_k such that $g_k = g(x_k), g'_k = g'(x_k), g''_k = g''(x_k)$, and $g'''_k = g'''(x_k)$. In general, the third derivative compact finite difference schemes can be written as:

$$\begin{aligned} & b_{-q}g'''_{k-q} + b_{-q+1}g'''_{k-q+1} + \cdots \\ & + b_{-1}g'''_{k-1} + b_0 g'''_k + b_1 g'''_{k+1} + \cdots + b_{p-1}g'''_{k+p-1} + b_p g'''_{k+p} \\ & = \frac{1}{h^3}\left[c_{-i}g_{k-i} + c_{-i+1}g_{k-i+1} + \cdots + c_{-1}g_{k-1} + c_0 g_k + c_1 g_{k+1} + \cdots \right. \\ & \left. + c_{i-1}g_{k+i-1} + c_i f_{k+i}\right] \end{aligned} \tag{1}$$

The polynomial of order N

$$F_N(x) = \sum_{r=0}^{N} d_r x^r \text{ and } F'''_N(x) = \sum_{r=3}^{N} r(r-1)(r-2)d_r x^{r-3} \tag{2}$$

can be applied to find the coefficients $b_{-q}, b_{-q+1}, \cdots, b_{p-1}, b_p, c_{-i}, c_{-i+1}, \cdots$, and c_i. In this paper, the numerical approximations of the third derivative can be constructed by applying the polynomial (2), in (1) as follows:

$$\begin{aligned} & b_{-q}\sum_{r=3}^{N} r(r-1)(r-2)d_r x_{k-q}^{r-3} + b_{-q+1}\sum_{r=3}^{N} r(r-1)(r-2)d_r x_{k-q+1}^{r-3} + \cdots \\ & + b_0 \sum_{r=3}^{N} r(r-1)(r-2)d_r x_k^{r-3} + \cdots + b_{p-1}\sum_{r=3}^{N} r(r-1)(r-2)d_r x_{k+p-1}^{r-3} \\ & + b_p \sum_{r=3}^{N} r(r-1)(r-2)d_r x_{k+p}^{r-3} \\ & = \frac{1}{h^3}\left(c_{-i}\sum_{r=0}^{N} d_r\left(x_{k-i}\right)^r + c_{-i+1}\sum_{r=0}^{N} d_r\left(x_{k-i+1}\right)^r + \cdots + c_0 \sum_{r=0}^{N} d_r\left(x_k\right)^r + \cdots \right. \\ & \left. + c_{i-1}\sum_{r=0}^{N} d_r\left(x_{k+i-1}\right)^r + c_i \sum_{r=0}^{N} d_r\left(x_{k+i}\right)^r\right) \end{aligned}$$

Since both x_k and h are arbitrary, the above equation can be simplified as follows.

$$\begin{aligned} & b_{-q}\sum_{r=3}^{N} r(r-1)(r-2)d_r(-q)^{r-3} + b_{-q+1}\sum_{r=3}^{N} r(r-1)(r-2)d_r(-q+1)^{r-3} + \cdots \\ & + 6b_0 d_3 + \cdots + b_{p-1}\sum_{r=3}^{N} r(r-1)(r-2)d_r(p-1)^{r-3} + b_p \sum_{r=3}^{N} r(r-1)(r-2)d_r(p)^{r-3} \\ & = \left(c_{-i}\sum_{r=0}^{N} d_r(-i)^r + c_{-i+1}\sum_{r=0}^{N} d_r(-i+1)^r + \cdots + c_0 d_0 + \cdots \right. \end{aligned}$$

$$+ c_{i-1}\sum_{r=0}^{N} d_r(i-1)^r + c_i\sum_{r=0}^{N} d_r(i)^r\Big) \tag{3}$$

The numerical scheme of order N satisfies all polynomials of degree $N-1$, so the equations below should be satisfied for the schemes in this work when $N = 0, 1,$ and 2:

$$\begin{aligned} \mathrm{c}_{-2} + \mathrm{c}_{-1} + \mathrm{c}_0 + \mathrm{c}_1 + \mathrm{c}_2 &= 0 \\ 2\mathrm{c}_{-2} + \mathrm{c}_{-1} - \mathrm{c}_1 - 2\mathrm{c}_2 &= 0 \\ 4\mathrm{c}_{-2} + \mathrm{c}_{-1} + \mathrm{c}_1 + 4\mathrm{c}_2 &= 0 \end{aligned}$$

When $N \geq 3$:

$$\begin{aligned} &\sum_{r=3}^{N} r(r-1)(r-2)d_r\big[b_{-q}(-q)^{r-3} + b_{-q+1}(-q+1)^{r-3} + \cdots + b_{p-1}(p-1)^{r-3} \\ &+ b_p(p)^{r-3}\big] + 6b_0d_3 \\ &= \sum_{r=3}^{N} d_r\left[c_{-i}(-i)^r + c_{-i+1}(-i+1)^r + \cdots + c_{i-1}(i-1)^r + c_i(i)^r\right] + c_0d_0 \end{aligned} \tag{4}$$

The degree of the polynomial $F_N(x)$ will decide the order of accuracy of the desired approximation by solving (4). This leads to the following.

$$\begin{aligned} -6\ \mathrm{b}_0 d_3 = \sum_{r=3}^{N} d_r\{&r(r-1)(r-2)\big[b_{-q}(-q)^{r-3} + b_{-q+1}(-q+1)^{r-3} + \cdots \\ &+ b_{p-1}(p-1)^{r-3} + b_p(p)^{r-3}\big] - \big[c_{-i}(-i)^r + c_{-i+1}(-i+1)^r + \cdots \\ &+ c_{i-1}(i-1)^r + c_i(i)^r\big]\} \end{aligned} \tag{5}$$

2.1. When q=0, p=0, i=2. :

In this case, the standard numerical scheme for the third derivative of order 4 can be constructed as follows:

$$\mathrm{b}_0 g_k''' = \frac{1}{h^3}\left[\mathrm{c}_{-2}g_{k-2} + \mathrm{c}_{-1}g_{k-1} + \mathrm{c}_0 g_k + \mathrm{c}_1 g_{k+1} + \mathrm{c}_2 g_{k+2}\right]$$

From (5), the leading equation is as follows:

$$-6\ \mathrm{b}_0 d_3 = \sum_{r=3}^{N} d_r\left[c_{-2}(-2)^r + c_{-1}(-1)^r + c_1 + c_2(2)^r\right]$$

When $N = 3$ and $N = 4$, the above equation gives:

$$\begin{aligned} -6\ \mathrm{b}_0 &= -8c_{-2} - c_{-1} + c_1 + 8c_2 \\ 0 &= 16c_{-2} + c_{-1} + c_1 + 16c_2 \end{aligned}$$

The system of this case can be written as follows:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 0 & 1 & 2 \\ 0 & 4 & 1 & 0 & 1 & 4 \\ -6 & -8 & -1 & 0 & 1 & 8 \\ 0 & 16 & 1 & 0 & 1 & 16 \end{bmatrix} \begin{bmatrix} b_0 \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system provides the values of the coefficients as follows.

$$b_0 = 1, c_{-2} = -0.5, c_{-1} = 1, c_0 = 0, c_1 = -1, \text{ and } c_2 = 0.5$$

2.2. When q=0, p=1, i=2 OR q=1, p=0, i=2. :

Compact numerical schemes for the third derivative of order 6 can be derived for these cases as follows:

2.2.1. When q=0, p=1, i=2. :

The scheme is: $\mathrm{b}_0 g_k''' + \mathrm{b}_1 g_{k+1}''' = \frac{1}{h^3}\left[\mathrm{c}_{-2} g_{k-2} + \mathrm{c}_{-1} g_{k-1} + \mathrm{c}_0 g_k + \mathrm{c}_1 g_{k+1} + \mathrm{c}_2 g_{k+2}\right]$

Equation (5) becomes:

$-6\ \mathrm{b}_0 d_3 = \sum_{r=3}^{N} d_r \left\{ r(r-1)(r-2)b_1 - \left[c_{-2}(-2)^r + c_{-1}(-1)^r + c_1 + c_2(2)^r\right]\right\}$

When $N = 3, 4, 5$, and 6 the system and its solution can be presented as follows:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 4 & 1 & 0 & 1 & 4 \\ -6 & -6 & -8 & -1 & 0 & 1 & 8 \\ 0 & -24 & 16 & 1 & 0 & 1 & 16 \\ 0 & -60 & -32 & -1 & 0 & 1 & 32 \\ 0 & -120 & 64 & 1 & 0 & 1 & 64 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$b_0 = 1, b_1 = 1, c_{-2} = 0, c_{-1} = -2, c_0 = 6, c_1 = -6$, and $c_2 = 2$

2.2.2. When q=1, p=0, i=2. :

The scheme can be written as:

$\mathrm{b}_{-1} g_{k-1}''' + \mathrm{b}_0 g_k''' = \frac{1}{h^3}\left[\mathrm{c}_{-2} g_{k-2} + \mathrm{c}_{-1} g_{k-1} + \mathrm{c}_0 g_k + \mathrm{c}_1 g_{k+1} + \mathrm{c}_2 g_{k+2}\right]$

Also, eq(5) turns into:

$-6\ \mathrm{b}_0 d_3$

$= \sum_{r=3}^{N} d_r \left\{ r(r-1)(r-2)b_{-1}(-1)^{r-3} - \left[c_{-2}(-2)^r + c_{-1}(-1)^r + c_1 + c_2(2)^r\right]\right\}$

Similarly, when $N = 3, 4, 5$, and 6 , the system and its results are given as

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 4 & 1 & 0 & 1 & 4 \\ -6 & -6 & -8 & -1 & 0 & 1 & 8 \\ 24 & 0 & 16 & 1 & 0 & 1 & 16 \\ -60 & 0 & -32 & -1 & 0 & 1 & 32 \\ 120 & 0 & 64 & 1 & 0 & 1 & 64 \end{bmatrix} \begin{bmatrix} b_{-1} \\ b_0 \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$b_{-1} = 1, b_0 = 1, c_{-2} = -2, c_{-1} = 6, c_0 = -6, c_1 = 2$, and $c_2 = 0$
It should be noted that similar schemes are founded for most cases. The cases $(q = 1, p = 1, i = 2), (q = 1, p = 1, i = 2), (q = 0, p = 2, i = 2), (q = 1, p = 2, i = 2)$ and $(q = 2, p = 1, i = 2)$ give the same scheme as in 2.2.1, while the case $(q = 2, p = 0, i = 2)$ implies the scheme of 2.2.2.

2.3. When q=2, p=2, i=2. :

A numerical approximation for the third derivative of order 8 is given from this case as follows:

$$\begin{aligned}&\mathrm{b}_{-2}g'''_{k-2} + \mathrm{b}_{-1}g'''_{k-1} + \mathrm{b}_0 g'''_k + \mathrm{b}_1 g'''_{k+1} + +\mathrm{b}_2 g'''_{k+2}\\ &= \frac{1}{h^3}\left[\mathrm{c}_{-2}g_{k-2} + \mathrm{c}_{-1}g_{k-1} + \mathrm{c}_0 g_k + \mathrm{c}_1 g_{k+1} + \mathrm{c}_2 g_{k+2}\right]\end{aligned}$$

From (5), this case can be represented as follows:

$$\begin{aligned}-6\ \mathrm{b}_0 d_3 = &\sum_{r=3}^{N} d_r \left\{ r(r-1)(r-2)\left[b_{-2}(-2)^{r-3} + b_{-1}(-1)^{r-3} + b_1 + b_2(2)^{r-3}\right]\right\}\\ &- \sum_{r=3}^{N} d_r \left\{ -\left[c_{-2}(-2)^r + c_{-1}(-1)^r + c_1 + c_2(2)^r\right]\right\}\end{aligned}$$

When $N = 3, 4, 5, 6, 7,$ and $8.$ the system and its solution are as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 1 & 4\\ -6 & -6 & -6 & -6 & -6 & -8 & -1 & 0 & 1 & 8\\ 48 & 24 & 0 & -24 & -48 & 16 & 1 & 0 & 1 & 16\\ -240 & -60 & 0 & -60 & -240 & -32 & -1 & 0 & 1 & 32\\ 960 & 120 & 0 & -120 & -960 & 64 & 1 & 0 & 1 & 64\\ -3360 & -210 & 0 & -210 & -3360 & -128 & -1 & 0 & 1 & 128\\ 16128 & -504 & 0 & -504 & -16128 & -512 & -1 & 0 & 1 & 512 \end{bmatrix} \begin{bmatrix} \mathrm{b}_{-2}\\ \mathrm{b}_{-1}\\ \mathrm{b}_0\\ \mathrm{b}_1\\ \mathrm{b}_2\\ \mathrm{c}_{-2}\\ \mathrm{c}_{-1}\\ \mathrm{c}_0\\ \mathrm{c}_1\\ \mathrm{c}_2 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}$$

$b_{-2} = 1/126, b_{-1} = 4/9, b_0 = 1, b_1 = 4/9, b_2 = 1/126, c_{-2} = -20/21, c_{-1} = 40/21, c_0 = 0,$
$c_1 = -40/21$, and $c_2 = 20/21$

3. Confirming The Accuracy For The Numerical Approximations

In addition to many important features of numerical methods, verifying their orders is an essential step to ensure more accurate results. In this paper, polynomials of the desired order $F_r(x) = d_r x^r$ are used in all schemes to compute the remaining terms. The scheme of 2.2.1 can be given as

$$g'''_k + g'''_{k+1} = \frac{1}{h^3}\left[-2g_{k-1} + 6g_k - 6g_{k+1} + 2g_{k+2}\right] \tag{6}$$

Employing $F_6(x) = d_6 x^6$ and $F'''_6(x) = 120 d_6 x^3$ in the above scheme with arbitrary valves of x_k and h leads to equal terms on both sides of 6 as follows:

The LHS of (6) is:

$$g'''_k + g'''_{k+1} = \left(120 d_6 x_k^3\right) + \left(120 d_6 x_{k+1}^3\right) = \left(120 d_6 x_k^3\right) + \left(120 d_6\left(x_k + h\right)^3\right) = 120 d_6 h^3$$

while the RHS of (6) is:

$$\frac{1}{h^3}\left[-2g_{k-1}+6g_k-6g_{k+1}+2g_{k+2}\right]$$
$$=\frac{1}{h^3}\left[-2d_6x_{k-1}^6+6d_6x_k^6-6d_6x_{k+1}^6+2d_6x_{k+2}^6\right]$$
$$=\frac{1}{h^3}\big[-2d_6\left(h^6-6h^5x_k+15h^4x_k^2-20h^3x^3+15h^2x_k^4-6hx_k^5+x_k^6\right)+6d_6x_k^6$$
$$-6d_6\left(h^6+6h^5x_k+15h^4x_k^2+20h^3x_k^3+15h^2x_k^4+6hx_k^5+x_k^6\right)$$
$$+2d_6\left(64h^6+192h^5x_k+240h^4x_k^2+160h^3x_k^3+60h^2x_k^4+12hx_k^5+x_k^6\right)\big]$$
$$=\frac{1}{h^3}\big[h^6\left(-2d_6-6d_6+128d_6\right)\big]=120d_6h^3$$

However, applying $F_7(x)$ and $F_7'''(x)$ in (7) generates error terms as follows. The LHS of (7) is:
$g_k'''+g_{k+1}'''=\left(210d_7x_k^4\right)+\left(210d_7x_{k+1}^4\right)=210d_7x_k^4+210d_7\left(x_k+h\right)^4=210h^4d_7$
while the RHS of (7) is:
$\frac{1}{h^3}\left[-2d_7\left(x_k-h\right)^7+6d_7x_k^7-6d_7\left(x_k+h\right)^7+2d_7\left(x_k+2h\right)^7\right]$

$$=\frac{1}{h^3}\left[-2d_7(-h)^7-6d_7(h)^7+2d_7(2h)^7\right]=\frac{1}{h^3}\left[h^7d_7(2-6+256)=252h^4d_7\right.$$

To obtain and illustrate the behavior of the error terms, the polynomials $F_{11}(x)$ and $F'''_{11}(x)$ are substituted for the introduced schemes. For the scheme in 2.1, the following equation ensures that the scheme has order 4 because d_i disappeared for all $i\leq 4$, and all remaining terms are multiplied by h^5 as follows:

$$E_r\left[O\left(h^5\right)\right]=g_k'''-\frac{1}{h^3}\left[-\frac{1}{2}g_{k-2}+g_{k-1}-g_{k+1}+\frac{1}{2}g_{k+2}\right]$$
$$=\frac{1}{h^3}\left[-6h^5\left(5d_5+21d_7+85d_9+341d_{11}\right)\right]$$

For the schemes in 2.2.1 and 2.2.2, the equations below verify that they are of order 6 because d_7 is the lowest appearing term from the error terms, which are multiplied by h^7 :

$$E_r\left[O\left(h^7\right)\right]=g_k'''+g_{k+1}'''-\frac{1}{h^3}\left[-2g_{k-1}+6g_k-6g_{k+1}+2g_{k+2}\right]$$
$$=\frac{1}{h^3}\left[-6h^7\left(7d_7+28d_8+86d_9+220d_{10}+517d_{11}\right)\right]$$
$$E_r\left[O\left(h^7\right)\right]=g_{k-1}'''+g_k'''-\frac{1}{h^3}\left[-2g_{k-2}+6g_{k-1}-6g_k+2g_{k+1}\right]$$
$$=\frac{1}{h^3}\left[6h^7\left(-7d_7+28d_8-86d_9+220d_{10}-517d_{11}\right)\right]$$

Finally, since the error terms for the scheme in 2.3 start with d_9 times h^9, the scheme is of order 8 as follows:

$$\begin{aligned} E_r\left[O\left(h^9\right)\right] = & \frac{1}{126}g'''_{k-2} + \frac{4}{9}g'''_{k-1} + g'''_k + \frac{4}{9}g'''_{k+1} + \frac{1}{126}g'''_{k+2} \\ & -\frac{1}{h^3}\left[-\frac{20}{21}g_{k-2} + \frac{40}{21}g_{k-1} - \frac{40}{21}g_{k+1} + \frac{20}{21}g_{k+2}\right] \\ & = \frac{1}{h^3}\left[\frac{80}{7}h^9\left(-d_9 + 88d_{11}\right)\right] \end{aligned}$$

4. Dispersion Analysis for The Numerical Approximations

Fourier analysis can be used as an excellent tool to determine the resolution of numerical schemes by measuring the dispersion and dissipation terms [19]. The dispersion for the proposed numerical approximations is investigated to see how well the introduced schemes can capture the wave propagation of different frequencies because the accurate numerical scheme should reproduce the wave propagation properties of the original differential equation. Dispersion errors are efficiently computed using the Fourier analysis with a wave number ω for the suggested schemes.

A semi-discrete conservative for the first derivative for a function g can be used when a conservative spatial derivative approximation is employed as follows:

$$\frac{du_j}{dt} = \frac{-\left(\hat{g}_{j+\frac{1}{2}} - \hat{g}_{j-\frac{1}{2}}\right)}{h} \tag{7}$$

where $\hat{g}$ is the numerical flux, so g_j can be expressed as

$$g_j = g\left(u\left(x_j, t\right)\right) = \frac{\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \hat{g}(\xi)d\xi}{h} \tag{8}$$

where h is the mish size in the x-direction, and H is the primitive function of $\hat{g}$ defined as

$$H_{j+\frac{1}{2}} = H\left(x_{j+\frac{1}{2}}\right) = \int_{-\infty}^{x_{j+\frac{1}{2}}} \hat{g}(\xi)d\xi = \sum_{i=-\infty}^{j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \hat{g}(\xi)d\xi = \sum_{i=-\infty}^{j} g_i h \tag{9}$$

So

$$g_j = \frac{\left(H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}\right)}{h}, H_{j+\frac{1}{2}} = \hat{g}_{j+\frac{1}{2}}, \text{ and } H_{j-\frac{1}{2}} = \hat{g}_{j-\frac{1}{2}}$$

Therefore,

$$g'\left(x_j\right) = g'_j = \frac{\left(\hat{g}_{j+\frac{1}{2}} - \hat{g}_{j-\frac{1}{2}}\right)}{h} = \frac{\left(H''_{j+\frac{1}{2}} - H'_{j-\frac{1}{2}}\right)}{h}$$

From Fourier analysis, the function g can be written as follows:

$$g(x) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{g}_k e^{\frac{2\pi i k x}{L}}$$

with the scaled wave number $w = \frac{2\pi k}{N}$ and $s = \frac{x}{h}$, it follows:

$$g(x) = g(sh) = \sum_{k=\frac{N}{2}}^{\frac{N}{2}} \hat{g}_k e^{iws} \tag{10}$$

The first derivative of (10) with respect to s can be determined as:

$$\mathrm{h}g'(sh) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} iw\hat{g}_k e^{iws}$$

Furthermore, the third derivative of (10) with respect to s can be calculated as:

$$h^3 g'''(sh) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} -iw^3\hat{g}_k e^{iws} \tag{11}$$

Substituting (10) and (11) into (6) results in:

$$h^3 \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{g}_k \left(iw^3 + iw^3 e^{iw}\right) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{g}_k \left(-2e^{-iw} + 6 - 6e^{iw} + 2e^{2iw}\right)$$

Solving for w using Euler's formula, $e^{iw} = \mathrm{Cos}[w] + i\,\mathrm{Sin}[w]$, gives that:

$$\omega = (2\,\mathrm{Sin}[k] - \mathrm{Sin}[2k])^{1/3}$$

With the same procedures as above, the wave numbers for schemes are given in sects. 2.2 and 2.3 are derived. For schemes in section 2.2:

$$\omega = 4\frac{(-2)^{1/3}\left(\mathrm{e}^{\frac{\mathrm{i}k}{2}}\sin\left[\frac{k}{2}\right]^3\right)^{1/3}}{\left(1+\mathrm{e}^{\mathrm{i}k}\right)^{1/3}}$$

For scheme in 2.3:

$$\omega = 4\frac{(-30)^{1/3}(-((-1+\mathrm{Cos}[k])\,\mathrm{Sin}[k]))^{1/3}}{(63+56\,\mathrm{Cos}[k]+\mathrm{Cos}[2k])^{1/3}}$$

The real parts of the modified wavenumbers, which represent the dispersions of the schemes, are compared with the exact wavenumber as shown in Fig.1 below.

The above figure clearly illustrates that the compact schemes (in sections 2.2 and 2.3) have a better ability to capture the third derivative for larger wavenumbers compared to the standard finite difference scheme in section 2.1. Also, the

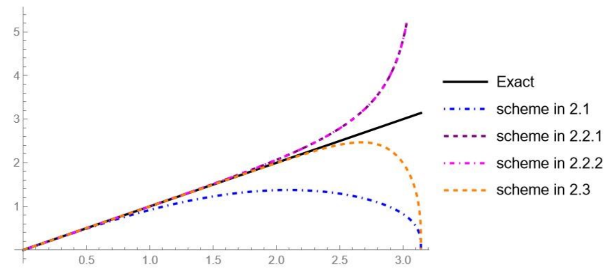

FIGURE 1. The dispersion terms of the introduced schemes

scheme in section 2.3 is the best among others because it is the closest to the exact wave number with less errors.

5. Conclusions

To approximate the third derivative, many high order schemes are constructed in this work by dealing with a class of linear systems. As a result of solving these systems, it leads to several kinds of numerical approximation with different orders of accuracy. To maintain stability and accuracy, two types of errors in this project are calculated using different approaches. First, error terms are computed by applying appropriate kinds of polynomials and matching both sides of the introduced schemes to obtain their exact orders of accuracy. Second, dispersive errors for the suggested schemes are checked by Fourier analysis to assure high resolutions ability of these schemes when applied to CFD problems with high frequency. From both error collection methods, fewer errors are generated by the presented schemes compared to standard finite difference methods. More schemes in desired orders can be derived from this work by considering more cases in which more points can be added to the construction equation.

Conflicts of interest : The authors declare that they have no conflict of interest.

Acknowledgments : The authors would like to thank Mustansirivah University, Baghdad, Iraq (www.uomustansiriyah.edu.iq) for its support in this work.

References

1. H. Abd Salman Al-Dujaly, M.A. Sabri, and S.Z. Hameed, *Some Results from the Upwinding Compact Scheme on Continuous and Non-continuous Functions*, Journal of Physics: Conference Series **1294** (2019), 032006.
2. H.A.S. Al-Dujaly, *An Approach of Finding Second Derivative Finite Differences Compact Schemes*, Iraqi Journal of Science **65** (2024), 5195-5206.
3. H.A.S. Al-Dujaly, and Y. Dong, *Matrix Form of Deriving High Order Schemes for the First Derivative*, Baghdad Science Journal **17** (2020), 1041-1048.

4. H.A.S. Al-Dujaly, Y. Yang, and C. Liu, *Weighted Upwinding Compact Scheme for Shock Capturing*, AIAA SciTech Forum, 55th AIAA Aerospace Sciences Meeting, 2017.
5. A.H. Ali, A.S. Jaber, M.T. Yaseen, M. Rasheed, O. Bazighifan, and T.A. Nofal, *A Comparison of Finite Difference and Finite Volume Methods with Numerical Simulations: Burgers Equation Mode*, Complexity **10** (2022), 9367638.
6. A. Bakheet, and E.A. Alnussairy, *Numerical Simulation of Magnetohydrodynamic Influences on Casson Model for Blood Flow Through an Overlapping Stenosed Artery*, Iraqi Journal of Science **62** (2021), 1016-1024.
7. J. Chen, and W. Dai, *A Compact Finite Difference Scheme with Absorbing Boundary Condition for Forced KdV Equation*, MethodsX **10** (2023), 102036.
8. W.J. Hasan, and K.A. Hussain, *Fifth Order Improved Runge-Kutta Nystrom Method Using Trigonometrically-Fitting for Solving Oscillatory Problem*, Al-Nahrain Journal of Science **25** (2022), 63-67.
9. H. Hani and M. Khenissi, *A New Two Derivative FSAL Runge–Kutta Method of Order Five in Four Stages*, Baghdad Science Journal **17** (2020), 166-171.
10. J.K. Kim, R. Verma, S. Dashputre, and Padmavati, *Convergence and Stability of Fibonacci SR-Iteration Process for Monotone Non-Lipschitzian Mapping*, Nonlinear Functional Analysis and Applications **30** (2025), 77-97.
11. K.S. Kim, *Convergence Theorems of Mixed Type Implicit Iteration for Nonlinear Mappings in Convex Metric Spaces*, Nonlinear Functional Analysis and Applications **27** (2022), 903-920.
12. M.A. Rasheed, A.T. Balasim, and A.F. Jameel, *Some results for the vorticity transport equation by using A.D.I scheme*, AIP Conference Proceedings **2138** (2019), 030031.
13. H. Ramos, A. Mehta, and G. Singh, *Compact Finite Difference Schemes with High Resolution Characteristics and Their Applications to Solve Burgers Equation*, Computational and Applied Mathematics **43** (2024), 109.
14. Rasheed, A. Maan, and Suad Naji Kadhim, *Numerical solutions of two-dimensional vorticity transport equation using Crank-Nicolson method*, Baghdad Science Journal **19** (2022), 23.
15. A.J. Saleh, L.J. Barghooth, and M.A. Rasheed, *Numerical Solutions of Nonlinear Fredholm Integro Differential Equations using Fifth Order Runge-Kutta Method*, Journal of Physics: Conference Series **1999** (2021), 012093.
16. L.V. Salian, S. Rathan, D. Ghosh, *A Novel Central Compact Finite-Difference Scheme for Third Derivatives with High Spectral Resolution*, arXiv preprint arXiv **2405** (2024), 00569.
17. H.A. Salman Al-Dujaly, A.T. Jassim, M.A. Sabri, and S.Z. Hameed, *The Construction and Analysis of Compact and Noncompact Schemes*, IOP Conference Series: Materials Science and Engineering, **571**, 2019, 012035.
18. Tan, Meiqi and Cheng, Juan and Shu, Chi-Wang, *Stability of High Order Finite Difference and Local Discontinuous Galerkin Schemes with Explicit-Implicit-Null Time-Marching for High Order Dissipative and Dispersive Equations*, Journal of Computational Physics **464** (2022), 111314.
19. R. Vichnevetsky, and J.B. Bowles, *Fourier Analysis of Numerical Approximations of Hyperbolic Equations*, SIAM, Philadelphia, 1982.
20. H.A. Wajid, N. Ahmed, H. Iqbal, and M.S. Arshad, *Modified Finite Difference Schemes on Uniform Grids for Simulations of the Helmholtz Equation at Any Wave Number*, Journal of Applied Mathematics **2014** (2014), 673106.
21. Z. Wang, H. Al-Dujaly, Y. Dong, and C. Liu, *Construction Methodology of Weighted Upwind Compact Scheme*, AIAA SciTech Forum, 54th AIAA Aerospace Sciences Meeting (2016), 2016-2061.
22. Wong, Yau and Li, Guangrui, *Exact Finite Difference Schemes for Solving Helmholtz Equation at Any Wavenumber*, International Journal of Numerical Analysis and Modeling. Series B **2** (2011), 91-108.

Hassan Abd Salman Al-Dujaly: Asst. Prof. at Mustansiriyah University. Received his Ph.D. from The University of Texas, USA in 2017. His research interests include Applied Mathematics, Numerical Analysis, and Applied Computational Mathematics.

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.
e-mail: hassanaldujaly@uomustansiriyah.edu.iq

Ahmed Bakheet: Asst. Prof. at Mustansiriyah University. Received his Ph.D. from Universiti Teknologi Malaysia in 2017. His research interests include Applied Mathematics, Fluid dynamics, and Numerical analysis.

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.
e-mail: ahmedbakheet@uomustansiriyah.edu.iq

Esam A. Alnussairy: Asst. Prof. at Wasit University. Received his Ph.D. from Universiti Teknologi Malaysia in 2017. His research interests include Applied Mathematics, Fluid dynamics, and Numerical analysis.

Department of Biology , College of Science, Wasit University, Iraq.
e-mail: eahmed@uowasit.edu.iq